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Parastatistics and the Lie Algebraical Methods Used There

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Abstract

We review some of the properties of the parafield operators and discuss in some detail where the difference between the ordinary and paraquantisation originates. Particular attention is paid to the many-vacuum representations of the para-Fermi operators.

1. *Introduction*

In this paper I would like to mention some properties of the parastatistics and in particular to draw the readers attention to the very close connection between the representations of the para-Fermi operators and the algebra of the orthogonal group.

To begin with I must mention that parastatistics was introduced by Green (1953) who observed that the commonly accepted rules of second quantisation, although sufficient, are not necessary however for satisfying all physical requirements, and pointed out how the quantisation axioms can be generalised. It is worth considering here in more detail the question of where the generalisation comes from. For simplicity, let us consider the quantisation on an example of a real scalar free field $\varphi(x)$.

2. *Classical Case*

One starts with the Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\varphi(x)\,.\,\partial^\mu\varphi(x) - \tfrac{1}{2}m^2\varphi^2(x) \tag{2.1}$$

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The Lagrange–Euler equation in this case is the Klein–Gordon equation with positive and negative frequency solutions $\varphi^{\pm}(x)$ of the form:

$$\varphi^{\pm}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\kappa}{\sqrt{(2\kappa^0)}} e^{\pm i\kappa x} \varphi^{\pm}(\boldsymbol{\kappa}), \qquad \kappa^0 = \sqrt{(\boldsymbol{\kappa}^2 + m^2)} \tag{2.2}$$

Noether's theorem, together with the invariance of $\mathscr{L}$ under the Poincaré group $\mathscr{P}$, gives the invariant quantities: the energy-momentum vector P^n and the angular-momentum tensor M^{em}. In particular, P^n may be represented as

$$P^n = \tfrac{1}{2} \int \mathbf{d}\mathbf{p} p^n [\varphi^+(\mathbf{p}), \varphi^-(\mathbf{p})]_+ \tag{2.3}$$

3. *Quantisation of the Field*

The quantisation can be performed in different equivalent ways; for instance, by postulating the equal-time commutation relations. For what we want to show, however, it is more convenient to follow the quantisation procedure given by Bogoliubov & Shirkov (1959). It is based on the following postulates:

(1) the field $\varphi(x)$ becomes an operator;

(2) the energy-momentum vector P and the angular-momentum tensor M are expressed in terms of the operator-field functions by the same expressions of the type (2.3), i.e., as in the classical case, with proper ordering of the operator factors.

It follows now from (1) and (2), together with the requirements that the field transforms according to a unitary representation of $\mathscr{P}$ and the compatibility of the transformation properties of the field and the state vectors, that $\varphi(x)$ satisfies the commutation relation

$$[P^n, \varphi^{\pm}(\boldsymbol{\kappa})] = \pm\kappa^n \varphi^{\pm}(\boldsymbol{\kappa}) \tag{3.1}$$

Inserting (2.3) and (3.1), we have

$$[[\varphi^+(\mathbf{p}), \varphi^-(\mathbf{p})]_+, \varphi^{\pm}(\boldsymbol{\kappa})] = \pm 2\delta(\mathbf{p} - \boldsymbol{\kappa})\varphi^{\pm}(\boldsymbol{\kappa}) \tag{3.2}$$

Generalising this equation, Green has postulated the three-linear structure relation for the parafields

$$[\varphi^+(\mathbf{q}), \varphi^+(\mathbf{p})]_{\pm}, \varphi^+(\boldsymbol{\kappa})] = 0$$
$$[[\varphi^+(\mathbf{q}), \varphi^-(\mathbf{p})]_{\pm}, \varphi^{\pm}(\boldsymbol{\kappa})] = \pm 2\delta(\mathbf{p} - \boldsymbol{\kappa})\varphi^{\pm}(\boldsymbol{\kappa}) \tag{3.3}$$

where the anticommutator (commutator) corresponds to the para-Bose (para-Fermi) field.

In the oridinary quantisation, one has to add two more postulates:

(3) the commutator or the anticommutator of two-field operators is a C number;

(4) all dynamical variables should be written in a normal product form.

From (3), it follows that

$$[\varphi^{+}(\mathbf{p}), \varphi^{+}(\boldsymbol{\kappa})] = [\varphi^{-}(\mathbf{p}), \varphi^{-}(\boldsymbol{\kappa})] = 0$$

and therefore the expression now corresponding to (3.2) is

$$\begin{aligned}[:[\varphi^{+}(\mathbf{p}), \varphi^{-}(\mathbf{p})]:, \varphi^{\pm}(\boldsymbol{\kappa})] &= 2[\varphi^{+}(\mathbf{p})\varphi^{-}(\mathbf{p}), \varphi^{\pm}(\boldsymbol{\kappa})] \\ &= 2\delta(\mathbf{p}-\boldsymbol{\kappa})\varphi^{\pm}(\boldsymbol{\kappa}) \end{aligned} \tag{3.2'}$$

Hence

$$[\varphi^{-}(\mathbf{p}), \varphi^{+}(\boldsymbol{\kappa})] = \delta(\mathbf{p}-\boldsymbol{\kappa}) \tag{3.3'}$$

and we obtain the Bose commutation relations. We see that the difference between the parastatistics and ordinary statistics is mainly due to postulate (3) which does not follow from some general physical requirements. However, the postulate greatly simplifies the practical calculations. So (modulo small details) we can conclude that the question of statistics depends mainly on whether the (anti) commutator of the fields is a C number (= ordinary statistics) or operator (= parastatistics).

Since the moment the parastructure relations were introduced, they were studied by several authors from different points of view. The general conclusion was, that in the real world as least, the known particles appear to satisfy only the ordinary statistics. Only at present the situation seems to be changing. There are indications that parastatistics may be relevant in physics and its abstract mathematical structure may have applications in some branches of mathematics. As an example, let us consider the prequantisation methods of Konstant–Souriau (1970). We have heard how these authors, reformulating in a proper way the physical quantisation procedure, go a long way towards constructing all irreducible unitary representations of connected Lie groups and, in particular, define all such representations for a large class of solvable groups. In this respect, it is interesting to ask whether the more general paraquantisation provides an extension of the prequantisation.

From a physical point of view the interest in parastatistics is now stimulated a great deal by the circumstance that the elementary particles seem to be formed out of constituents as, for example, in the coloured quark model (Fritzsch & Gell-Mann, 1971). In this model, whose predictions are in reasonable agreement with experiment, the quarks seem to behave as para-Fermions of order 3, as first pointed out by Greenberg (1964).

The order of the parastatistics is a concept which characterises a certain class of representations of the para-operators (PO), namely the Fock-type representations. One way to obtain a p-order representation is to express the para-operators in terms of another set of p operators, namely (Green, 1953)

$$\varphi^{\pm}(\mathbf{p}) = \sum_{i=1}^{p} \varphi^{i\pm}(\mathbf{p}) \tag{3.4}$$

where the so-called Green ansatz $\varphi^{i\pm}(\mathbf{p})$ satisfy anomalous commutation relations:

$$[\varphi^{i\pm}(\mathbf{p}), \varphi^{j\pm}(\mathbf{q})]_\epsilon = 0 \qquad i \neq j$$
$$[\varphi^{i-}(\mathbf{p}), \varphi^{i+}(\mathbf{q})]_{-\epsilon} = \delta(\mathbf{p}-\mathbf{q})$$
$$[\varphi^{i-}(\mathbf{p}), \varphi^{i-}(\mathbf{q})]_{-\epsilon} = [\varphi^{i+}(\mathbf{p}), \varphi^{i+}(\mathbf{q})]_{-\epsilon} = 0 \tag{3.5}$$

The sign $\epsilon = +(-)$ corresponds to the para-Bose (para-Fermi) case. Let ϕ_p be the Fock space of the Green ansatz with vacuum $|0\rangle$. This space, in general, is reducible with respect to PO. The representation of order p is realised in the irreducible subspace $W_p \subset \phi_p$ of PO, which contains $|0\rangle$. The vector $|0\rangle$ is the only state in W_p which is annihilated by $\varphi^-(\mathbf{p})$ and therefore these representations contain a unique vacuum state. All representations of this type were found and classified by Greenberg & Messiah (1965) and in the context we shall refer to them as canonical representations. The space ϕ_p, however, contains also PO irreducible subspaces with more than one linearly independent state vanishing under the action of the annihilation operators $\varphi^-(\mathbf{p})$, i.e., in these subspaces the vacuum is degenerate. Previously such representations were rejected on the ground of the uniqueness of the vacuum. The concept of the vacuum, however, depends on the physical meaning ascribed to the operators involved. If one accepts the attitude that the elementary particles have no internal structure, then the vacuum should be the only state annihilated by $\varphi^-(\mathbf{p})$, and hence one is forced to admit only canonical representations. On the other hand, if the particles are assumed to have internal structure, as for instance to be composed of quarks, it is quite natural to demand the vacuum to be unique with respect to the constituents. As an example let us assume that the field is of parastatistics 3. Then, one can write

$$\varphi^{\pm}(x) = \frac{1}{(2\pi)^{3/2}} \sum_{i=1}^{3} \int \frac{d\mathbf{p}}{\sqrt{(2p^0)}}\, e^{\pm ipx} \varphi^{i\pm}(\mathbf{p}) \tag{3.6}$$

and demand that the representation space contains only one state annihilated by the constituent annihilation operators $\varphi^{i-}(\mathbf{p})$. This means that one has to consider as representation space the Fock space ϕ_3 of the Green ansatz. The space ϕ_3 is irreducible with respect to $\varphi^{i\pm}(\mathbf{p})$ and, as we have already observed, contains several states annihilated by the field $\varphi^-(x)$. The vacuum is, however, unique. This point of view was first suggested by Govorkov (1968) and was properly realised in a recent paper of Bracken & Green (1973). The former authors propose to consider generalised parastatistics of order p, which makes use of all representations of PO in ϕ_p. The vectors from ϕ_p, which are annihilated by $\varphi^-(\mathbf{p})$, are called in this case reservoir or vacuum-like states.

So we conclude that not only the canonical representations may be relevant for physical applications. Therefore, we proceed to study all representations of PO and, in particular, to analyse the properties of these representations with respect to the reservoir states. We shall, however, restrict our considerations only to the case of para-Fermi operators (PFO). The reason for this is a purely technical one. It is due to the circumstance that to determine all PFO representations we make essential use of the underlying Lie algebraical structure of the PFO, namely of the fact that every irreducible representation of n pairs of PFO can be extended to an irreducible representation of the classical Lie algebra B_n and vice versa. This allows us in the case of PFO to formulate the whole problem in a purely Lie algebraical language, whereas this seems to be impossible for the para-Bose case.

For later use, we first consider a finite number $a_i, b_i, i \in N = (j \mid j = 1, \ldots, n)$ of n pairs of para-Fermi operators. The full set of relations they satisfy is the following:

$$
\begin{aligned}
[[a_p, b_q], a_r] &= 2\delta_{qr} a_p \\
[[a_p, b_q], b_r] &= -2\delta_{pr} b_q \\
[[b_p, b_q], a_r] &= 2(\delta_{qr} b_p - \delta_{pr} b_q) \\
[[a_p, a_q], b_r] &= 2(\delta_{qr} a_p - \delta_{pr} a_q) \\
[[a_p, a_q], a_r] &= [[b_p, b_q], b_r] = 0
\end{aligned} \tag{3.7}
$$

Let T be the associative free algebra of $a_i, b_i, i \in N$, and J be the two-sided ideal in T generated by the relations (3.7). The factor algebra $U = T/J$ is called para-Fermi algebra. It is an infinite dimensional Lie algebra with respect to the commutator $[x, y] = xy - yx, x, y \in U$. To find a representation of the PFO means to find a representation of the associative algebra U in some Hilbert (or, more generally, in some linear) space. Denote by B_n the subspace of all antisymmetrised second-order polynomials of PFO in U. From (3.7) one can easily see that B_n is a Lie subalgebra of the para-Fermi algebra and also that U is the universal enveloping algebra of B_n. Thus the para-Fermi algebra coincides with the universal enveloping algebra of B_n. Since, by definition, a representation of B_n is a homomorphism of U in an algebra of operators in some Hilbert space, we conclude the following.

Proposition 1. Every representation of the PFO is a representation of B_n. The representation of the para-Fermi operators is irreducible if and only if the representation of B_n is irreducible.

Thus the problem of constructing all representations of PFO is reduced to a Lie algebraical one. Let us note that one is interested in considering not all representations but only those for which the Hermitian conjugate of the annihilation operator a_i equals b_i. Therefore, one needs to determine all representations of U considered as a $*$-algebra with involution defined by the condition $(a_i)^* = b_i, i \in N$. The set of all irreducible representations of the operators $a_i, b_i, i \in N$ is now completely determined from the observation that as a complex algebra B_n is isomorphic to the classical algebra denoted in the same way (Ryan & Sudarshan, 1963), whereas, as a real algebra, it is so$(n, n+1)$ (Palev, 1972). Moreover, the condition $(a_i)^* = b_i$ may be fulfilled only for the finite dimensional representations of B_n which are known (Gel'fand & Zeithlin, 1950). It is convenient to choose the basis in B_n in the following way:

$$\omega_i = \frac{[a_i, b_i]}{8n-4}$$

$$e_{\omega_i} = a_i, \qquad e_{-\omega_i} = b_i$$

$$e_{\omega_i+\omega_j} = [a_i, a_j], \qquad e_{-\omega_i-\omega_j} = [b_i, b_j]$$

$$e_{\omega_i-\omega_j} = [a_i, b_j] \tag{3.8}$$

The generators $[a_i, b_j]$, $i, j \in N$, span a basis for a subalgebra of B_n isomorphic to $U(n)$. In the following, when considering $U(n)$, we shall understand this particular realisation. The vectors $\Omega = (\omega_1, \ldots, \omega_n)$ are orthogonal with respect to the Cartian–Killing form, namely

$$(\omega_i, \omega_j) = \frac{\delta_{ij}}{4n-2} \tag{3.9}$$

and can be taken as a basis of the Cartan subalgebra of B_n. In this case the root system Σ is given by the vectors

$$\Sigma = (\pm\omega_i \pm \omega_j, \pm\omega_i \mid i, j \in N) \tag{3.10}$$

Hence the weights corresponding to $a_i(b_i)$ have positive first non-vanishing coordinate in Ω. Therefore, we obtain

Proposition 2. The Cartan subalgebra of B_n can be chosen in such a way that all para-Fermi annihilation (creation) operators $a_i(b_i)$, $i \in N$, are positive (negative) root vectors.

Let us turn now to the representations of PFO. Consider an irreducible representation realised in a space W and let x_Λ be the highest weight of the representation with weight

$$\Lambda = \sum_{i=n}^{n} L_i\omega_i \equiv (L_1, \ldots, L_n)$$

As is known, the coordinates of Λ, which are all non-negative (half) integers such that $L_1 \geqslant \cdots \geqslant L_n$, define the representation of B_n and hence of PFO up to an isomorphism. A vector $(l_1, \ldots, l_n)$ is a weight if and only if it has (half) integer coordinates satisfying the inequality

$$\sum_{\kappa=1}^{m} |l_{i_\kappa}| \leqslant \sum_{\kappa=1}^{m} L_\kappa \forall i_1 \neq i_2 \neq \cdots \neq i_m \in N, \qquad m = 1, \ldots, n \tag{3.11}$$

Relation (3.9), together with the equality $\omega_i . x_\Lambda = (\Lambda, \omega_i)x_i$, gives that $\frac{1}{2}[a_i, b_i]x_\Lambda = L_i x_\Lambda$ so that L_i are the eigenvalues of $\frac{1}{2}[a_i, b_i]$ on the highest weight vector.

Proposition 2 greatly simplifies the problem of finding all reservoir states contained in W. We recall that $x \in W$ is said to be a reservoir state if $a_i . x = 0 \; \forall i \in N$. Let $P \subset U$ be the set of all polynomials of the $U(n)$ generators. Denote by $V \subset W$ the $U(n)$ representation space, which contains x_Λ, namely $V = (px_\Lambda \,|\, p \in P)$.

Proposition 3. The vector $x \in W$ is a reservoir state if and only if $x \in V$. The subspace V is the linear envelope of all weight vectors with weights $(l_1, \ldots, l_n)$ such that

$$\sum_{i=1}^{n} l_i = \sum_{i=1}^{n} L_i \tag{3.12}$$

Omitting the details of the proof [see Palev (1973)] we observe that the weight vector x_l with weight $l = (l_1, \ldots, l_n)$ satisfying (3.12) is necessarily annihilated by all a_i since otherwise the corresponding weight to $a_i x_l$ would violate the condition (3.11). It is also clear from the construction that V carries an irreducible $U(n)$ representation. Consider now the important special case of representation with highest weight $\Lambda = (L, L, \ldots, L)$ of x_Λ. Clearly, for $i \neq j$ $[a_i, b_j]x_\Lambda$ is not a weight vector and therefore it is zero. Combining this with the formula for $i = j$ and taking into account that $a_i . x_\Lambda = 0 \; \forall \; i \in N$ we obtain:

$$a_i b_j . x_\Lambda = \delta_{ij} p x_\Lambda \tag{3.13}$$

where we have put $p = 2L$.

It follows from (3.13) that $V = Px_\Lambda = x_\Lambda$ and therefore we obtain in this case a representation with a single reservoir state, i.e., a canonical representation of PFO. In fact equation (3.13) is the defining relation for representation of parastatistics p. So we see that the canonical representations of PFO are kinds of most degenerate representations of B_n. The representation of n pairs of usual Fermi operators corresponds, for instance, to a highest weight $(\frac{1}{2}, \ldots, \frac{1}{2})$. It is also easy to show that only the representations with $L_1 = \cdots = L_n$ contain a single reservoir state.

Let us now examine in more detail the reservoir states in an irreducible representation of a countable number $a_i, b_i, i \in I = (j \,|\, j = 1, 2, \ldots)$ of PFO. We define, as before, the para-Fermi algebra and the algebra B of all second-order antisymmetrised polynomials. In this case B is an infinite dimensional Lie algebra, and the para-Fermi algebra $U(B)$ is its universal enveloping algebra in complete analogy with the finite case. The elements $p \in U(B)$ are polynomials of finite number of PFO.

Consider an irreducible representation of the PFO in some Hilbert space W and let $x_0 \in W$ be a reservoir state, i.e., $a_i . x_0 = 0 \; \forall \; i \in I$. From the irreducibility it follows that the subspace $U_0 = (px_0 \,|\, p \in U(B))$ is dense in W. Let

$x_1 = p_0 x_0, p_0 \in U(B)$ be another reservoir state. Since p_0 is a polynomial of finite, say, n para-Fermi operators, we can put without loss of generality that p_0 is a function of $a_i, b_i, i \in N$. The representation of $a_i, b_i, i \in N$ in W can in general be reducible. From the full reducibility of the representations of B_n we have that x_0 can be represented in a unique way as a sum $x_0 = \Sigma_\alpha x_0^\alpha$, where x_0^α belongs to the B_n irreducible subspace W^α of W, and the subspaces W^α are linearly independent. Since $a_i . x_0^\alpha \in W^\alpha, i \in N, a_i . x_0 = 0$ implies that $a_i . x_0^\alpha = 0 \ \forall \ \alpha$ and $i \in N$. Hence x_0^α is a B_n reservoir state. On the other hand $a_i p_0 x_0^\alpha \in W^\alpha, i \in N$ and therefore $a_i p_0 x_0 = 0$ if and only if $a_i p_0 x_0^\alpha = 0$ for all α and $i \in N$. Applying now Proposition 3 we have that $a_i p_0 x_0^\alpha = 0, i \in N$, only in the case where p_0 is a polynomial of $[a_i, b_j], i, j \in N$. On the other hand, let p_0 be a polynomial of $[a_i, b_j], i, j \in N$. For $K \notin N$ $[a_K, [a_i, b_j]] = 0$ and hence $[a_K, p_0] = 0$. Therefore, $a_K p_0 x_0 = p_0 a_K x_0 = 0$ and $p_0 x_0$ is a reservoir state. We have proved that for a given reservoir state x_0 the vector $p_0 x_0$ is also a reservoir state if and only if p_0 is a polynomial of $[a_i, b_j]$, $i, j \in I$. The set of all such elements spans a basis of an infinite dimensional Lie algebra A, which is a subalgebra of B. Denote by $U(A)$ its universal enveloping algebra, $U(A) \subset U(B)$. We have now that all reservoir states are contained in the subspace $V \subset W$, which is the closure of all vectors px_0, $p \in U(A)$. The algebras B and A satisfy the commutation relations of the orthogonal and unitary algebras correspondingly, and therefore they can be considered as infinite dimensional analogue of the classical algebras B_n and A_n. The representation of A in V is an irreducible one. Indeed by construction, x_0 is a cyclic vector of A in V. Let $x_1 \in U(A)x_0$. Interchanging the places of x_0 and x_1 in the above considerations, we find that there exists an operator $p \in U(B)$ such that $x_0 = px_1$. Moreover, since x_0 and x_1 are reservoir states, $p \in U(A)$ and therefore $U(A)$ contains a projection operator on x_0 for any vector from a dense set in V. Repeating the same argument for x_0 and any other vector x_2 from the dense set in V, we obtain that for any two vectors x_1, x_2 from the dense set of V, there exists an operator $q \in U(A)$ such that $x_2 = qx_1$. Hence the representation of A in V is irreducible. In fact, we have proved the Haag–Schroer (1962) lemma. So we have:

Proposition 4. Any irreducible representation of the algebra B defines an irreducible representation of an infinite number $a_i, b_i, i \in I$ of PFO and vice versa. All reservoir states from the representation space W of PFO are contained in a subspace $V \subset W$ which carries an irreducible representation of the algebra A.

In conclusion, let me stress that the question whether parastatistics exists in nature depends mainly on whether the field (anti) commutators are C numbers. The requirement that the field (anti) commutators be C numbers does not follow from any general physical requirements. If parastatistics exists, then the kind of representation of PO operators to be used depends on the physical meaning ascribed to these operators. It is worth mentioning that the extension to parastatistics preserves the essential properties of several of the field-theoretical models. For instance, one can easily verify that in any current algebra model the generalisation to parastatistics preserves the commutation relations between the currents. This is due to the fact that the currents can always be expressed as functions of $[a_i, b_j]_\epsilon$ with $\epsilon = +(-)$ for the Bose (Fermi) case (Adler & Dashen, 1968). It could be interesting to observe that the Bose and para-Fermi statistics can be viewed as quite similar mathematical structures. The Bose operators are generators of a solvable Lie algebra, the Heisenberg algebra, whereas the PFO are generators of the simple Lie algebra of the rotation group. In both cases, the dynamical variables are functions of Lie algebra generators.

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